Marginal distribution of the S-matrix elements for Dyson's measure and some applications

1983 J. Phys. A: Math. Gen. 16 237

(http://iopscience.iop.org/0305-4470/16/2/007)

J. Phys. A: Math. Gen. 16 1983) 237–254. Printed in Great Britain

Marginal distribution of the S-matrix elements for Dyson's measure and some applications

P Pereyra†§ and P A Mello‡§

† Departamento de Ciencias Básicas, Universidad Autónoma Metropolitana, Atzcapotzalco, México

‡ Instituto de Física, UNAM, México, and Instituto Nacional de Investigaciones Nucleares, México

Received 10 June 1982

Abstract. In recent attempts to construct a statistical theory of nuclear reactions by doing statistics directly on the S-matrix elements, Dyson's measure, which remains invariant under an automorphism that maps the space of unitary and symmetric matrices into itself, is of fundamental importance. In the present paper, we study some of the marginal distributions of the individual S-matrix elements, or of groups of them, that arise from Dyson's measure. To understand the problem better, a similar discussion is first carried out for Haar's measure of unitary matrices which do not have the restriction of symmetry, and some of the effects of this restriction are thus exhibited. Some applications of these mathematical results to reaction theory are discussed.

1. Introduction

In the traditional statistical theory of nuclear reactions (Kawai *et al* 1973, Moldauer 1975, 1978, Hofmann *et al* 1975, Agassi *et al* 1975) one constructs ensembles of scattering matrices S in terms of more 'microscopic' quantities, like the matrix elements of the Hamiltonian, the poles and residues of a K-matrix, etc. Recently, there have been attempts to do statistics directly on the S-matrix elements, by proposing a measure in the space of scattering matrices (Mello 1979, Mello and Seligman 1980, De los Reyes *et al* 1980, Hofmann *et al* 1981). The S-matrix is unitary because of flux conservation and, if the problem is time-reversal invariant, S is also symmetric. The first step is to define a measure that gives equal *a priori* probability to all unitary and symmetric matrices of a given order n. This intuitive notion can be defined precisely by asking for a measure $d\mu(S)$ that remains invariant under the automorphism

$$S \to \tilde{S} = U^0 S U^{0\mathrm{T}}, \tag{1.1}$$

that maps the space of unitary and symmetric matrices into itself. Here U^0 is an arbitrary unitary matrix and T denotes transposition. Dyson (1962) has shown that the condition

$$d\mu(S) = d\mu(U^0 S U^{0\mathrm{T}}) \tag{1.2}$$

§ Address: Instituto de Física, UNAM, Apartado postal 20-364, Delegación Alvaro Obregón, 01000 México, DF, Mexico.

0305-4470/83/020237 + 18\$02.25

defines the measure uniquely (up to a constant factor); $d\mu(S)$ will be referred to as *Dyson's measure.*

To make an analogy with the field of statistical mechanics, Dyson's measure is the equivalent of the volume element in phase space, which assigns equal *a priori* probabilities in that space and remains invariant under canonical transformations. In both fields, ensembles that contain more information than the invariant one are then constructed by multiplying the latter by appropriate functions of S, or of the coordinates and momenta, respectively.

Since Dyson's measure can be considered as the building block for the construction of more complicated ensembles, it is clearly important to study the distribution of individual S-matrix elements or groups of them that arise from such a measure. That study is the main purpose of the present paper. The results that we have been able to obtain are presented in § 3.1: specifically, we succeeded in calculating the joint distribution of any number of S-matrix elements in one row, and that of the four matrix elements contained in a 2×2 block along the diagonal, like $\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$.

The condition of symmetry on the S-matrix complicates the various calculations considerably. One can learn a great deal by studying a simplified problem first: that of an ensemble of unitary matrices U, without the restriction of symmetry, governed by the invariant measure of the unitary group, i.e. Haar's measure $dh(U)$, which remains invariant under the operation

$$U \to \tilde{U} = U^0 U \qquad \text{or} \qquad \tilde{U} = UU^0, \tag{1.3a, b}$$

i.e.

$$dh(U) = dh(U^0 U) = dh(UU^0). \tag{1.4}$$

Although many properties of Haar's measure for the unitary group are well known, we first present in § 2.1 a brief analysis of some of the marginal distributions arising from $dh(U)$, since some of the results will be needed in the study of the S-matrix carried out in the following section; some other results will be used for comparison with the corresponding ones for the S-matrix, in order to understand better the effect of the symmetry requirement.

In §§ 2.2 and 3.2 we discuss some applications to reaction theory of the mathematical results contained in §§ 2.1 and 3.1. The applications presented in § 3.2 are relevant to nuclear physics, whereas those of § 2.2 are not, because of the lack of symmetry, and are presented here as a useful exercise (they could be relevant to a system subject to a very strong magnetic field). The reader interested only in the mathematical aspects of the problem can skip §§ 2.2 and 3.2, since §§ 2.1 and 3.1 are self-contained.

2. The problem of unitary matrices U

2.1. *The joint marginal probability density for the U matrix elements*

As was outlined in the Introduction, we study in this section an ensemble of unitary matrices U of order n, not restricted by the condition of symmetry, and governed by Haar's measure $dh(U)$ of (1.4).

We write

$$U_{ab} = x_{ab} + iy_{ab}. \tag{2.1}$$

Consider a set of k matrix elements $U_{a_1b_1}, \ldots, U_{a_kb_k}$ and call $dp_0(U_{a_1b_1}, \ldots, U_{a_kb_k})$

the probability assigned by Haar's measure to the volume element $dx_{a_1b_1}\, dy_{a_1b_1} \ldots dx_{a_kb_k}\, dy_{a_kb_k}$. We define the joint marginal probability density $p_0(U_{a_1b_1}, \ldots, U_{a_kb_k})$ associated with these variables by the relation

$$dp_0(U_{a_1b_1}, \ldots, U_{a_kb_k}) = p_0(U_{a_1b_1}, \ldots, U_{a_kb_k})\, dx_{a_1b_1}\, dy_{a_1b_1} \ldots dx_{a_kb_k}\, dy_{a_kb_k}. \tag{2.2}$$

Specifically, we shall analyse in what follows the joint probability density for the elements of one and two rows. The joint probability density $p_0(U_{11}, \ldots, U_{1n})$ of the elements of, say, the first row contains the delta function $\delta(1-\Sigma_{a=1}^{n} |U_{1a}|^2)$. Moreover, p_0 must remain invariant under *any* transformation of the type (1.3*b*), since any such transformation only mixes the elements of one row among themselves, but not with other rows. We should thus multiply the δ-function by an arbitrary function of $U_{11}, \ldots, U_{1n}$ that remains invariant under (1.3*b*). Since the only invariant is the norm $\Sigma_{a=1}^{n} |U_{1a}|^2$, the arbitrary function reduces to a constant, due to the δ-function. We thus have

$$p_0(U_{11}, \ldots, U_{1n}) \propto \delta\left(1 - \sum_{a=1}^{n} |U_{1a}|^2\right). \tag{2.3}$$

By integrating equation (2.3) over the variables $U_{1,k+1}, \ldots, U_{1n}$ we find, for the joint probability density of the elements $U_{11}, \ldots, U_{1k}$, the expression

$$p_0(U_{11}, \ldots, U_{1k}) \propto \left(1 - \sum_{a=1}^{k} |U_{1a}|^2\right)^{n-k-1}, \qquad n > k. \tag{2.4}$$

Notice that (2.4) remains invariant under the operation (1.3*b*), with U^0 consisting of two blocks, of dimensionalities k and $n-k$, in order not to mix $U_{11}, \ldots, U_{1k}$ with $U_{1,k+1}, \ldots, U_{1n}$. Consider now the joint probability density of, say, the first and second rows, associated with Haar's measure. It can be written as

$$p_0\begin{pmatrix} U_{11} \ldots U_{1n} \\ U_{21} \ldots U_{2n} \end{pmatrix} \propto \delta\left(1 - \sum_{a=1}^{n} |U_{1a}|^2\right) \delta\left(1 - \sum_{a=1}^{n} |U_{2a}|^2\right) \delta\left(\sum_{a=1}^{n} U_{1a} U_{2a}^*\right), \tag{2.5}$$

because of the normalisation of the two rows and their orthogonality. If we want to keep only the first k elements of each row, we have to integrate over the $2(n-k)$ remaining ones. Defining the complex vectors

$$\boldsymbol{r}_1 \equiv (U_{11}, \ldots, U_{1k}), \qquad \boldsymbol{r}_2 \equiv (U_{21}, \ldots, U_{2k}), \tag{2.6}$$

their joint probability density is found in appendix 1 to be (ϑ is the usual step function)

$$p_0(\boldsymbol{r}_1, \boldsymbol{r}_2) \propto [1 - \boldsymbol{r}_1 \cdot \boldsymbol{r}_1^* - \boldsymbol{r}_2 \cdot \boldsymbol{r}_2^* + (\boldsymbol{r}_1 \cdot \boldsymbol{r}_1^*)(\boldsymbol{r}_2 \cdot \boldsymbol{r}_2^*) - |\boldsymbol{r}_1 \cdot \boldsymbol{r}_2^*|^2]^{n-k-2} \vartheta(1 - \boldsymbol{r}_1^2) \vartheta(1 - \boldsymbol{r}_2^2). \tag{2.7}$$

Consider again the particular case of the transformation (1.3*b*), with U^0 consisting of two blocks, of dimensionalities k and $n-k$; such a transformation conserves the norms of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ and their scalar product, so that (2.7) remains invariant.

For the particular case $k = 2$, which will be needed later on, equation (2.7) gives

$$p_0\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \propto (1 - |U_{11}|^2 - |U_{12}|^2 - |U_{21}|^2 - |U_{22}|^2 + |U_{11}U_{22} - U_{12}U_{21}|^2)^{n-4}$$
$$\times \vartheta(1 - |U_{11}|^2 - |U_{12}|^2) \vartheta(1 - |U_{21}|^2 - |U_{22}|^2). \tag{2.8}$$

Notice the appearance of the absolute value of the determinant $U_{11}U_{22} - U_{12}U_{21}$, which also remains invariant under the transformation considered above.

2.2. Applications

We first analyse the large-n limit of the above results, since this limit is important for many applications. Consider the probability of U_{11}, given by equation (2.4) with $k=1$. If we expand its logarithm, we obtain

$$p_0(U_{11}) \propto (1-|U_{11}|^2)^{n-2} \underset{n\gg 1}{\approx} \exp(-n|U_{11}|^2) = \exp(-nx_{11}^2)\exp(-ny_{11}^2), \tag{2.9}$$

confirming a conjecture made by Gaudin and Mello (1981). The order of magnitude of x_{11}^2 and y_{11}^2 is n^{-1}. The next term in the expansion of $\ln p_0(U_{11})$, nx_{11}^4, is $O(n^{-1})$ and is thus neglected. We thus see that for $n \gg 1$, x_{11} and y_{11} are distributed according to independent zero-centred Gaussians, with equal variances given by $(2n)^{-1}$.

The elastic fluctuation cross section $\sigma_{11}^{\mathrm{fl}}$ is defined as $\langle|U_{11}-\langle U_{11}\rangle_0|^2\rangle$ (see Kawai *et al* 1973), so that

$$\sigma_{11}^{\mathrm{fl}} = \operatorname{var} x_{11} + \operatorname{var} y_{11} \approx 1/n, \qquad n \gg 1, \tag{2.10}$$

which coincides with the result found by Gaudin and Mello (1981).

Similarly, the large-n limit of $p_0(U_{11}, U_{12})$, obtained from (2.4) with $k=2$, is

$$p_0(U_{11}, U_{12}) \propto \exp(-n|U_{11}|^2)\exp(-n|U_{12}|^2), \qquad n \gg 1, \tag{2.11}$$

that corresponds to two independent zero-centred Gaussians, with the same variance $1/n$. Equation (2.9) follows from (2.11) by integration. With the same procedure, the $n \gg 1$ limit of (2.8) corresponds to four independent zero-centred Gaussians with variance $1/n$.

We now apply the results of § 2.1 to the ensemble of U matrices defined by Gaudin and Mello (1981), which is of maximum entropy, subject to the constraint $\langle U\rangle$ = fixed, the basic measure being given by $\mathrm{d}h(U)$. The differential probability associated with that ensemble is given by

$$\mathrm{d}p(U) = \exp(-\mathrm{Re}\,\mathrm{Tr}\,\beta U)\,\mathrm{d}h(U) \Big/ \int \exp(-\mathrm{Re}\,\mathrm{Tr}\,\beta U')\,\mathrm{d}h(U'). \tag{2.12}$$

Here β is a matrix of Lagrange multipliers that allows fixing of $\langle U\rangle$.

The above results allow the study of the distribution of the variables $U_{11}, \{U_{11}, U_{12}\}$, $\{U_{11}, U_{12}, U_{21}, U_{22}\}$ in some particular cases. We start with $p(U_{11})$. Consider the problem in which the average of all U_{ab} is kept zero except that of U_{11} which, for convenience, is taken to be real (Gaudin and Mello 1981), with an *arbitrary* value between -1 and $+1$, i.e.

$$\bar{x}_{11} \equiv \langle x_{11}\rangle \neq 0, \qquad \langle y_{11}\rangle = 0, \qquad \langle U_{12}\rangle = \ldots = \langle U_{nn}\rangle = 0, \tag{2.13}$$

where we have used the notation of (2.1).

From (2.12) and (2.4) with $k=1$, we can immediately write $p(U_{11})$ as

$$p(U_{11}) \propto \exp(-\beta x_{11})(1-|U_{11}|^2)^{n-2}. \tag{2.14}$$

The large-n limit of $p(U_{11})$ of equation (2.14) cannot be obtained by simply replacing the second factor in equation (2.14) by the Gaussian of equation (2.9), since (2.14) is now peaked at a value $\bar{x}_{11} \neq 0$ (notice that this naive replacement would give variances (and hence $\sigma_{11}^{\mathrm{fl}}$) independent of the centroid $\bar{x}_{11}$); we must therefore approximate the second factor in (2.14) for values of x_{11} in the vicinity of $\bar{x}_{11}$. Writing

$$x_{11} = \bar{x}_{11} + \xi_{11} \tag{2.15}$$

in (2.14), we have

$$p(U_{11}) \propto \exp(-\beta x_{11})(1-x_{11}^2-y_{11}^2)^{n-2}$$
$$= \exp(-\beta x_{11})[1-(\bar{x}_{11})^2]^{n-2}\left(1-\frac{2\bar{x}_{11}\xi_{11}+\xi_{11}^2+y_{11}^2}{1-(\bar{x}_{11})^2}\right)^{n-2}$$
$$\propto \exp\left[-\beta x_{11}+(n-2)\ln\left(1-\frac{2\bar{x}_{11}\xi_{11}+\xi_{11}^2+y_{11}^2}{1-(\bar{x}_{11})^2}\right)\right]. \tag{2.16}$$

Expanding the ln up to second order in ξ_{11} and y_{11}, we have

$$p(U_{11}) \propto \exp\left(-2n\frac{1+(\bar{x}_{11})^2}{[1-(\bar{x}_{11})^2]^2}\frac{\xi_{11}^2}{2}-\frac{2n}{1-(\bar{x}_{11})^2}\frac{y_{11}^2}{2}\right). \tag{2.17}$$

We have cancelled the linear term in ξ_{11} with the choice

$$\beta = 2n\bar{x}_{11}/[1-(\bar{x}_{11})^2]. \tag{2.18}$$

Notice that we are keeping $\bar{x}_{11}$ fixed and arbitrary between -1 and $+1$, and taking the limit $n \gg 1$. The next term in the expansion of the exponent in (2.16) is $O(n^{-1})$ and is thus neglected. Within this approximation, in the present case of total absorption in all channels and arbitrary absorption in channel 1, ξ_{11} and y_{11} are thus statistically independent, they have a Gaussian distribution centred at $\bar{x}_{11}$ and 0, respectively, and variances given by

$$\operatorname{var} x_{11} = \frac{1}{2n}\frac{[1-(\bar{x}_{11})^2]^2}{1+(\bar{x}_{11})^2}, \qquad \operatorname{var} y_{11} = \frac{1-(\bar{x}_{11})^2}{2n}. \tag{2.19}$$

The fluctuation cross section σ_{11}^{fl} is now

$$\sigma_{11}^{\text{fl}} = \operatorname{var} x_{11} + \operatorname{var} y_{11} = [1-(\bar{x}_{11})^4]^{-1}P_1^2/n, \tag{2.20}$$

where we have introduced the transmission factor (Kawai *et al* 1973)

$$P_c = 1-|\langle U_{cc}\rangle|^2. \tag{2.21}$$

That both variances in (2.19) depend on $\bar{x}_{11}$ is a consequence of the way in which ξ_{11} and y_{11} appear in the second factor of (2.14) which, in turn, arises from the normalisation of the first row (unitarity). That $\operatorname{var} x_{11}$ decreases as $\bar{x}_{11}$ increases can be seen very clearly in the simple case of the distribution of x_{11} alone. From (2.4) we have

$$p_0(x_{11}) \propto (1-x_{11}^2)^{n-3/2}, \tag{2.22}$$

so that from (2.14)

$$p(x_{11}) \propto \exp(-\beta x_{11})(1-x_{11}^2)^{n-3/2}. \tag{2.23}$$

We saw that if the Gaussian approximation is used for $p_0(x_{11})$, $p(x_{11})$ is a shifted Gaussian with the same width. However, the exact expression for $p_0(x_{11})$ tends to zero more rapidly than the Gaussian approximation, since it has to vanish at $|x_{11}| = 1$, with the net effect of narrowing the distribution resulting from the product (2.23).

In a similar way, again under conditions (2.13), the distribution of U_{11} and U_{12} is obtained from (2.4) (with $k=2$) and (2.12) as

$$p(U_{11}, U_{12}) \propto \exp(-\beta x_{11})(1-|U_{11}|^2-|U_{12}|^2)^{n-3} \tag{2.24a}$$

$$\underset{n\gg 1}{\approx} \exp -\left[2n\frac{1+(\bar{x}_{11})^2}{1-(\bar{x}_{11})^2}\frac{\xi_{11}^2}{2}+\frac{2n}{1-(\bar{x}_{11})^2}\frac{y_{11}^2}{2}+\frac{2n}{1-(\bar{x}_{11})^2}\left(\frac{x_{12}^2}{2}+\frac{y_{12}^2}{2}\right)\right], \tag{2.24b}$$

which is again Gaussian in the $n \gg 1$ limit. Equation (2.17) follows upon integrating (2.24) over U_{12}. The fluctuation cross sections σ_{11}^{fl} and σ_{12}^{fl} are now given by

$$\sigma_{11}^{\text{fl}} = [1-(\bar{x}_{11})^4]^{-1}P_1P_1/n, \qquad \sigma_{12}^{\text{fl}} = P_1P_2/n. \tag{2.25}$$

Recall that $P_1 \neq 1$, $P_2 = \ldots = P_n = 1$; equations (2.25) are then in the form of a Hauser–Feshbach (1952) expression, $P_aP_b/\mathrm{Tr}\,P$, with a correction factor in the elastic case. We notice the interesting fact that the correction factor, which is 1 for total absorption, is quite close to unity up to, say, $\bar{x}_{11}=0.5$, where it takes the value 1.07. This is therefore the same property for the so-called elastic enhancement factor W, which is defined as the ratio of the correction factors for the elastic and inelastic cases.

Finally, we consider the distribution of U_{11}, U_{12}, U_{21} and U_{22}. We shall analyse this in the case when the averages of U_{11} and U_{22} are different from zero and real, while all the others are taken to be zero, i.e.

$$\bar{x}_{11} \equiv \langle x_{11}\rangle \neq 0, \langle y_{11}\rangle = 0, \qquad \bar{x}_{22} \equiv \langle x_{22}\rangle \neq 0, \langle y_{22}\rangle = 0,$$
$$\langle U_{33}\rangle = \ldots = \langle U_{nn}\rangle = 0, \qquad \langle U_{ab}\rangle = 0, a \neq b.$$

From (2.8) and (2.12) we have

$$p\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22}\end{pmatrix} \propto \exp(-\beta_1 x_{11})\exp(-\beta_2 x_{22})$$
$$\times(1-|U_{11}|^2-|U_{12}|^2-|U_{21}|^2-|U_{22}|^2+|U_{11}U_{22}-U_{12}U_{21}|^2)^{n-4}. \tag{2.26}$$

The large-n limit of this expression is obtained again by expanding the above expression around the mean values of the variables involved. We make the substitution

$$x_{11} = \bar{x}_{11} + \xi_{11}, \qquad x_{22} = \bar{x}_{22} + \xi_{22}, \tag{2.27}$$

take the logarithm of (2.26) and keep up to quadratic terms in ξ_{11}, y_{11}, ξ_{22}, y_{22}, x_{12}, y_{12}, x_{21}, y_{21}. The linear terms are again cancelled by an appropriate choice of β_1 and β_2. The result is

$$p\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22}\end{pmatrix} \propto \exp -\left(n\frac{1+(\bar{x}_{11})^2}{[1-(\bar{x}_{11})^2]^2}\xi_{11}^2 + \frac{n}{1-(\bar{x}_{11})^2}y_{11}^2 + n\frac{1+(\bar{x}_{22})^2}{[1-(\bar{x}_{22})^2]^2}\xi_{22}^2\right.$$
$$+\frac{n}{1-(\bar{x}_{22})^2}y_{22}^2 + \frac{n}{[1-(\bar{x}_{11})^2][1-(\bar{x}_{22})^2]}$$
$$\left.\times[x_{12}^2+y_{12}^2+x_{21}^2+y_{21}^2+2\bar{x}_{11}\bar{x}_{22}(x_{12}x_{21}-y_{12}y_{21})]\right), \tag{2.28}$$

which corresponds again to Gaussian variables, but now U_{12} and U_{21} are no longer independent. The bilinear expression can be diagonalised to find the variances, with the result for the fluctuation cross sections

$$\sigma_{11}^{\text{fl}} = \frac{1}{1-(\bar{x}_{11})^4}\frac{P_1P_1}{n}, \qquad \sigma_{12}^{\text{fl}} = \frac{1}{1-(\bar{x}_{11}\bar{x}_{22})^2}\frac{P_1P_2}{n},$$
$$\sigma_{21}^{\text{fl}} = \frac{1}{1-(\bar{x}_{11}\bar{x}_{22})^2}\frac{P_2P_1}{n}, \qquad \sigma_{22}^{\text{fl}} = \frac{1}{1-(\bar{x}_{22})^4}\frac{P_2P_2}{n}, \tag{2.29}$$

which correspond again to Hauser–Feshbach expressions with correction factors that, for $\bar{x}_{11}$ and $\bar{x}_{22}$ up to 0.5, do not differ appreciably from unity.

3. The problem of unitary and symmetric matrices S

3.1. The joint marginal probability density for the S matrix elements

We now address ourselves to the real purpose of this paper, which is the study of an ensemble of unitary and symmetric matrices S of order n, governed by Dyson's measure $d\mu(S)$ of (1.2). We write

$$S_{ab} = X_{ab} + iY_{ab} \tag{3.1}$$

and define a probability density just as we did in (2.2), replacing U by S and x, y by X, Y.

Specifically, we have been able to study the joint probability density for the elements of one row and that for the 2×2 block $\{S_{11}, S_{12}, S_{21}, S_{22}\}$. This we present in what follows.

Consider, say, the first row, and designate by $p_0(S_{11}, \ldots, S_{1n})$ the corresponding probability density. From unitarity, p_0 contains the delta function $\delta(1-\Sigma_{a=1}^{n}|S_{1a}|^2)$. Consider all those transformations of the type (1.1) which do not mix $S_{11}, \ldots, S_{1n}$ with the other S-matrix elements: those transformations should keep p_0 invariant, because of the defining property (1.2). In contrast to the previous section, this does not happen now for any transformation (1.1), but only for a restricted class: functions of the corresponding invariants will now appear in the expression for p_0. We shall determine this class. Equation (1.1) can be written as

$$\tilde{S}_{ab} = \sum_{\alpha\beta} U_{a\alpha}^0 U_{b\beta}^0 S_{\alpha\beta} \tag{3.2a}$$

and, for the first row,

$$\tilde{S}_{1b} = \sum_{\alpha\beta} U_{1\alpha}^0 U_{b\beta}^0 S_{\alpha\beta}. \tag{3.2b}$$

If the RHS of (3.2b) has to contain $S_{1\beta}$ only, then

$$U_{1\alpha}^0 = e^{i\theta_1}\delta_{1\alpha}. \tag{3.3}$$

The matrix U^0 will then have the structure

$$\mathcal{U} = \begin{pmatrix} e^{i\theta_1} & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & \boxed{\mathcal{U}^0} & \\ 0 & & & \end{pmatrix}. \tag{3.4}$$

The transformed matrix elements (3.2b) are then

$$\tilde{S}_{11} = e^{2i\theta_1} S_{11}, \qquad \tilde{S}_{1b} = e^{i\theta_1} \sum_{\beta=2}^{n} \mathcal{U}^0_{b\beta} S_{1\beta}, \qquad b \neq 1. \tag{3.2c}$$

Since $\mathcal{U}^0$ is unitary (with dimensionality $n-1$), functions of $S_{11}, \ldots, S_{1n}$ that remain invariant under (3.2c) are

$$|\tilde{S}_{11}|^2 = |S_{11}|^2, \qquad \sum_{b=2}^{n} |\tilde{S}_{1b}|^2 = \sum_{b=2}^{n} |S_{1b}|^2. \tag{3.5}$$

But the last quantity coincides with $1-|S_{11}|^2$, from unitarity. Our distribution p_0 can then be written as

$$p_0(S_{11}, \ldots, S_{1n}) \propto f(|S_{11}|^2)\,\delta\Big(1 - \sum_{b=1}^{n} |S_{1b}|^2\Big). \tag{3.6}$$

We have no way to specify f further just from invariance properties. However, it turns out that the probability density for S_{11} alone can be found independently, and f can then be calculated therefrom. We now see how this can be done.

Every unitary and symmetric S can be written as

$$S = UU^{\mathrm{T}}, \tag{3.7}$$

where U is unitary. The transformation (1.1) on S,

$$\tilde{S} = U^0(UU^{\mathrm{T}})U^{0\mathrm{T}} = (U^0U)(U^0U)^{\mathrm{T}} = \tilde{U}\tilde{U}^{\mathrm{T}} \tag{3.8a}$$

with

$$\tilde{U} \equiv U^0 U, \tag{3.8b}$$

can then be interpreted as the transformation (1.3) on U. We can thus write the following relation between Dyson's and Haar's measures (Mello and Seligman 1980):

$$\mathrm{d}\mu(S(U)) = \mathrm{d}h(U) \tag{3.9}$$

Therefore, the distribution of S can be found in terms of that of U. In particular, to find the probability density $p_0(S_{11})$ we realise that, since

$$S_{11} = \sum_{a=1}^{n} (U_{1a})^2, \tag{3.10}$$

all we need is the joint probability density for the elements of one row of U, which was given in equation (2.3). We therefore have

$$p_0(S_{11}) \propto \int \ldots \int \delta\Big(S_{11} - \sum_{a=1}^{n} (U_{1a})^2\Big)\,\delta\Big(1 - \sum_{a=1}^{n} |U_{1a}|^2\Big)\,\mathrm{d}U_{11} \ldots \mathrm{d}U_{1n},$$

$$\mathrm{d}U_{ab} \equiv \mathrm{d}x_{ab}\,\mathrm{d}y_{ab}, \qquad U_{ab} = x_{ab} + \mathrm{i}y_{ab}. \tag{3.11}$$

The details of the calculation are given in appendix 2, where the following result is found:

$$p_0(S_{11}) \propto (1 - |S_{11}|^2)^{(n-3)/2}. \tag{3.12}$$

This result has a structure similar to the one found for unitary matrices, equation (2.4) with $k=1$, except for the value of the exponent.

On the other hand, $p_0(S_{11})$ can be obtained from (3.6) by integrating over $S_{12}, \ldots, S_{1n}$, i.e.

$$p_0(S_{11}) \propto f(|S_{11}|^2) \int \delta\Big(1 - \sum_{a=1}^{n} |S_{1a}|^2\Big)\, \mathrm{d}S_{12} \ldots \mathrm{d}S_{nn}$$

$$\propto f(|S_{11}|^2)(1-|S_{11}|^2)^{n-2}. \tag{3.13}$$

Comparing (3.12) and (3.13), we find $f(|S_{11}|^2)$ as

$$f(|S_{11}|^2) \propto (1-|S_{11}|^2)^{(1-n)/2}, \tag{3.14}$$

so that the joint distribution (3.6) for the first row becomes

$$p_0(S_{11}, \ldots, S_{1n}) \propto (1-|S_{11}|^2)^{(1-n)/2} \delta\Big(1 - \sum_{a=1}^{n} |S_{1a}|^2\Big). \tag{3.15}$$

In contrast to equation (2.3), where all the elements of a given row appear on the same footing, the factor $f(|S_{11}|^2)$ distinguishes now between the diagonal and off-diagonal matrix elements.

Several interesting consequences can now be obtained from equation (3.15). Integrating (3.15) over the variables $S_{1,k+1}, \ldots, S_{1n}$ we find, for the joint probability density of the elements $S_{11}, \ldots, S_{1k}$, the expression

$$p_0(S_{11}, \ldots, S_{1k}) \propto (1-|S_{11}|^2)^{(1-n)/2} \Big(1 - \sum_{a=1}^{k} |S_{1a}|^2\Big)^{n-k-1}, \qquad n > k. \tag{3.16}$$

Integrating (3.16) over S_{11} we find, for the joint probability density of the non-diagonal elements $S_{12}, \ldots, S_{1k}$,

$$p_0(S_{12}, \ldots, S_{1k}) \propto \Big(1 - \sum_{a=2}^{k} |S_{1a}|^2\Big)^{n-k}$$

$$\times {}_2F_1\Big(\frac{n-1}{2}, 1; n-k+1; 1 - \sum_{a=2}^{k} |S_{1a}|^2\Big), \qquad n > k, \tag{3.17}$$

where ${}_2F_1$ is the usual hypergeometric function. The asymmetry between diagonal and non-diagonal matrix elements is again apparent from the above results (3.16) and (3.17).

The probability density of the element S_{12} can be obtained from (3.17) putting $k = 2$:

$$p_0(S_{12}) \propto (1-|S_{12}|^2)^{n-2} {}_2F_1((n-1)/2, 1; n-1; 1-|S_{12}|^2), \qquad n > 2. \tag{3.18}$$

From the symmetry properties of Dyson's measure, we can now say that equation (3.12) gives the probability density of any diagonal S-matrix element and equation (3.18) that of any non-diagonal element.

The distribution $p_0(S_{12})$ for $n = 2$ can be obtained directly from (3.15) for $n = 2$, by integrating over S_{11}, with the result

$$p_0(S_{12}) \propto 1/|S_{12}|, \qquad n = 2. \tag{3.19}$$

Use of the relation

$${}_2F_1(\tfrac{1}{2}, 1; 1; z) = (1-z)^{-1/2} \tag{3.20}$$

shows that (3.18) is also valid for $n = 2$.

From the above results it is easy to find some of the moments of the various distributions, which turn out to be important for several applications. From (3.16) with $k=2$ one can obtain, by direct integration, the crossed moments of the variables S_{11} and S_{12}:

$$\langle |S_{11}|^{l} |S_{12}|^{m} \rangle_0 = \frac{\Gamma(n)\Gamma(m/2+1)\Gamma(l/2+1)\Gamma((n+m-1)/2)}{2\Gamma(m/2+n-1)\Gamma((l+n+m+1)/2)}. \tag{3.21}$$

The crossed moments of the variables S_{12}, S_{13} are easily found by integrating $|S_{12}|^{l}|S_{13}|^{m}$ times the joint probability density $p_0(S_{11}, S_{12}, S_{13})$ obtained from (3.16) with $k=3$, with the result

$$\langle |S_{12}|^{l} |S_{13}|^{m} \rangle_0 = \frac{\Gamma(l/2+1)\Gamma(m/2+1)\Gamma(n)}{\Gamma(n+(l+m)/2-1)(n+l+m-1)}. \tag{3.22}$$

Equations (3.21) and (3.22) can be seen to agree with the particular cases that were calculated by Mello and Seligman (1980).

We now generalise the above analysis to find the joint probability density of the block of four S-matrix elements $\{S_{11}, S_{12}, S_{21}, S_{22}\}$. From (3.7) it is clear that we now need the joint probability density of two rows of U, which was given in equation (2.5). The calculation is sketched in appendix 3, the result being

$$p_0\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \propto \delta(S_{12}-S_{21})(1-|S_{11}|^2-|S_{22}|^2-2|S_{12}|^2+|S_{11}S_{22}-(S_{12})^2|^2)^{(n-5)/2}. \tag{3.23}$$

This equation has a structure similar to that of equation (2.8) for unitary matrices, except for the value of the exponent and the appearance of the delta function $\delta(S_{12}-S_{21})$. The reason for this similarity is not clear at present; a proper understanding of it would probably simplify the calculation of the S-matrix distributions.

It is clear that $p_0(\begin{smallmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{smallmatrix})$ should remain invariant under those transformations (1.1) in which U^0 is of the type

$$U^0 = \begin{pmatrix} u_{11} & u_{12} & 0 & \dots & 0 \\ u_{21} & u_{22} & & & \\ 0 & & & & \\ \vdots & & & \mathcal{U}^0 & \\ 0 & & & & \end{pmatrix}, \tag{3.24}$$

where the matrices

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \tag{3.25}$$

and $\mathcal{U}^0$ are, respectively, 2×2 and $(n-2)\times(n-2)$ unitary matrices. In fact, the absolute value of the determinant $S_{11}S_{22}-(S_{12})^2$ remains invariant under the transformation

$$\tilde{s} = usu^{\mathrm{T}} \tag{3.26}$$

where

$$s \equiv \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

Noticing that we can write

$$(\tilde{S}_{11}\tilde{S}_{12}\tilde{S}_{21}\tilde{S}_{22})^{\mathrm{T}} = u \times u(S_{11}S_{12}S_{21}S_{22})^{\mathrm{T}} \tag{3.27}$$

where $\times$ indicates the usual direct product, we see that the norm $|S_{11}|^2+|S_{12}|^2+|S_{21}|^2+|S_{22}|^2$ is also invariant. As a result, the distribution p_0 of equation (3.23) remains invariant, too.

3.2. *Applications*

We first analyse the large-n limit of the above results, since in many practical cases one deals with many open channels.

For $n \gg 1$, $p_0(S_{11})$ of equation (3.12) gives

$$p_0(S_{11}) \propto (1-|S_{11}|^2)^{(n-3)/2} \approx \exp(-\tfrac{1}{2}n|S_{11}|^2) = \exp(-\tfrac{1}{2}nX_{11}^2)\exp(-\tfrac{1}{2}nY_{11}^2), \tag{3.28}$$

showing that in this limit X_{11} and Y_{11} become independent zero-centred Gaussian variables, with the same variance $1/n$. The fluctuation cross section $\sigma_{11}^{\mathrm{fl}}$ is thus given by

$$\sigma_{11}^{\mathrm{fl}} = \langle|S_{11}-\langle S_{11}\rangle_0|^2\rangle_0 = \operatorname{var} X_{11} + \operatorname{var} Y_{11} \approx 2/n, \tag{3.29}$$

which differs by a factor 2 from the corresponding expression (2.10) for U.

Similarly, the large-n limit of $p_0(S_{11}, S_{12})$, obtained from (3.16) with $k=2$, is

$$p_0(S_{11}, S_{12}) \propto (1-|S_{11}|^2)^{(n-5)/2}\left(1-\frac{|S_{12}|^2}{1-|S_{11}|^2}\right)^{n-3} \approx \exp(-\tfrac{1}{2}n|S_{11}|^2)\exp(-n|S_{12}|^2), \tag{3.30}$$

so that S_{11} and S_{12} become independent zero-centred Gaussian variables, in agreement with the results of Agassi *et al* (1975). The fluctuation cross sections are given by

$$\sigma_{11}^{\mathrm{fl}} = \operatorname{var} S_{11} = 2/n, \qquad \sigma_{12}^{\mathrm{fl}} = 1/n, \tag{3.31}$$

exhibiting in a clear fashion the effect of $f(|S_{11}|^2)$ (equation (3.16)) in generating the elastic enhancement factor $W=2$.

With the same procedure, the $n \gg 1$ limit of (3.23) is (3.30) multiplied by the Gaussian $\exp[-(n/2)|S_{22}|^2]$ and the delta function $\delta(S_{12}-S_{21})$.

We now apply the results of § 3.1 to the ensemble of S-matrices that was defined by Mello (1979) and Mello and Seligman (1980) as the one of maximum entropy, subject to the constraint $\langle S\rangle$ = fixed, the basic measure being $\mathrm{d}\mu(S)$. The differential probability associated with that ensemble is given by

$$\mathrm{d}p(S) = \frac{\mathrm{e}^{-\mathrm{Re}\,\mathrm{Tr}\,\beta S}\,\mathrm{d}\mu(S)}{\int \mathrm{e}^{-\mathrm{Re}\,\mathrm{Tr}\,\beta S'}\,\mathrm{d}\mu(S')}, \tag{3.32}$$

where β is a matrix of Lagrange multipliers that allows the fixing of $\langle S\rangle$.

The above results allow us to study the distribution of the variables S_{11}, $\{S_{11}, S_{12}\}$, $\{S_{11}, S_{12}, S_{21}, S_{22}\}$ in some particular cases. Consider the problem in which the average of all S_{ab} is kept zero, except that of S_{11} which, for convenience, will be taken real, with an arbitrary value between -1 and 1, i.e.

$$\bar{X}_{11} \equiv \langle X_{11}\rangle \neq 0, \qquad \langle Y_{11}\rangle = 0, \qquad \langle S_{12}\rangle = \ldots = \langle S_{nn}\rangle = 0. \tag{3.33}$$

From (3.32), (3.12) and (3.16) (with $k=2$), we thus have

$$p(S_{11}) \propto \exp(-\beta X_{11})(1-|S_{11}|^2)^{(n-3)/2}, \tag{3.34}$$

$$p(S_{11}, S_{12}) \propto \exp(-\beta X_{11})(1-|S_{11}|^2)^{(1-n)/2}(1-|S_{11}|^2-|S_{12}|^2)^{n-3}. \tag{3.35}$$

The large-n limit of these expressions is

$$p(S_{11}) \underset{n\gg 1}{\propto} \exp\left(-n\frac{1+(\bar{X}_{11})^2}{[1-(\bar{X}_{11})^2]^2}\frac{\xi_{11}^2}{2}-\frac{n}{1-(\bar{X}_{11})^2}\frac{Y_{11}^2}{2}\right), \tag{3.36}$$

$$p(S_{11}, S_{12}) \underset{n\gg 1}{\propto} \exp\left(-n\frac{1+(\bar{X}_{11})^2}{[1-(\bar{X}_{11})^2]^2}\frac{\xi_{11}^2}{2}-\frac{n}{1-(\bar{X}_{11})^2}\frac{Y_{11}^2}{2}-\frac{2n}{1-(\bar{X}_{11})^2}\frac{|S_{12}|^2}{2}\right), \tag{3.37}$$

where $\xi_{11} \equiv X_{11}-\bar{X}_{11}$. The fluctuation cross sections can now be written as

$$\sigma_{11}^{\text{fl}} = \frac{2}{1-|\langle S_{11}\rangle|^4}\frac{P_1P_1}{n}, \qquad \sigma_{12}^{\text{fl}} = \frac{P_1P_2}{n}, \tag{3.38}$$

where P_c is the transmission factor (Kawai *et al* 1973)

$$P_c = 1-|\langle S_{cc}\rangle|^2. \tag{3.39}$$

That ξ_{11}, Y_{11} and S_{12} are Gaussians in equation (3.37) agrees with the results of Agassi *et al* (1975).

We now consider the distribution of the four matrix elements S_{11}, S_{12}, S_{21} and S_{22}. This we shall analyse in the case when the averages of S_{11} and S_{22} are different from zero and real, and arbitrary between -1 and $+1$, while all the others are taken to be zero, i.e.

$$\bar{X}_{11} \equiv \langle X_{11}\rangle \neq 0, \qquad \langle Y_{11}\rangle = 0, \qquad \bar{X}_{22} \equiv \langle X_{22}\rangle \neq 0, \qquad \langle Y_{22}\rangle = 0$$
$$\langle S_{33}\rangle = \ldots = \langle S_{nn}\rangle = 0, \qquad \langle S_{ab}\rangle = 0, \qquad a \neq b. \tag{3.40}$$

From (3.23) and (3.32) we have

$$p\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22}\end{pmatrix} \propto \exp(-\beta X_{11})\exp(-\beta X_{22})$$
$$\times[1-(|S_{11}|^2+|S_{22}|^2+2|S_{12}|^2)+|S_{11}S_{22}-(S_{12})^2|]^{(n-5)/2}\,\delta(S_{12}-S_{21}). \tag{3.41}$$

This expression can be approximated in the large-n limit by expanding the logarithm around the mean values (3.40) (where $\bar{X}_{11}$ and $\bar{Y}_{11}$ are kept fixed, with *arbitrary* values between -1 and $+1$) just as was done in going from (2.26) to (2.28), with the result

$$p\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22}\end{pmatrix} \propto \exp-\left(\frac{n}{2}\frac{1+(\bar{X}_{11})^2}{[1-(\bar{X}_{11})^2]^2}\xi_{11}^2+\frac{n}{2}\frac{1}{1-(\bar{X}_{11})^2}Y_{11}^2\right.$$
$$+\frac{n}{2}\frac{1+(\bar{X}_{22})^2}{[1-(\bar{X}_{22})^2]^2}\xi_{22}^2+\frac{n}{2}\frac{1}{1-(\bar{X}_{22})^2}Y_{22}^2$$
$$\left.+n\frac{(1+\bar{X}_{11}\bar{X}_{22})X_{12}^2+(1-\bar{X}_{11}\bar{X}_{22})Y_{12}^2}{[1-(\bar{X}_{11})^2][1-(\bar{X}_{22})^2]}\right)\delta(S_{12}-S_{21}), \tag{3.42}$$

showing that the real and imaginary parts of S_{11} and S_{22} are independent Gaussian variables. The resulting fluctuation cross sections are

$$\sigma_{11}^{\mathrm{fl}}=\frac{2}{1-|\bar{S}_{11}|^4}\frac{P_1P_1}{n},\qquad \sigma_{12}^{\mathrm{fl}}=\frac{1}{1-|\bar{S}_{11}|^2|\bar{S}_{22}|^2}\frac{P_1P_2}{n},$$
$$\sigma_{21}^{\mathrm{fl}}=\frac{1}{1-|\bar{S}_{11}|^2|\bar{S}_{22}|^2}\frac{P_2P_1}{n},\qquad \sigma_{22}^{\mathrm{fl}}=\frac{2}{1-|\bar{S}_{22}|^4}\frac{P_2P_2}{n}. \tag{3.43}$$

It is interesting to compare (3.43) with the result of Agassi *et al* (1975), which is also valid in the large-n limit. The difference between the two results is the extra factors $f_{ab}=(1-|\bar{S}_{aa}|^2|\bar{S}_{bb}|^2)^{-1}$ that appear in the present treatment. We notice, just as we did in the previous section, that even for $\bar{S}_{11}$ and $\bar{S}_{22}$ as large as 0.5, this factor does not differ appreciably from unity ($f=1.07$).

Appendix 1. Proof of equation (2.7)

The joint probability density for the first k elements of say, the first and second rows, can be obtained by integration of equation (2.5) over the $2(n-k)$ remaining elements. To begin, we write the two-rows joint probability density as

$$p_0\begin{pmatrix}U_{11},\ldots,U_{1n}\\U_{21},\ldots,U_{2n}\end{pmatrix}=\delta(1-|\boldsymbol{R}_1|^2)\delta(1-|\boldsymbol{R}_2|^2)\delta(\boldsymbol{R}_1\cdot\boldsymbol{R}_2)\delta(\boldsymbol{R}_1\cdot\boldsymbol{M}\boldsymbol{R}_2) \tag{A1.1}$$

where

$$\boldsymbol{R}_a=(x_{a1},y_{a1},\ldots,x_{an},y_{an}) \tag{A1.2}$$

and

$$\boldsymbol{M}=\begin{pmatrix}0&-1&0&&\cdots&&0\\1&0&&&&&\\0&&0&-1&&&\\&&1&0&&&\\\vdots&&&&\ddots&&\\&&&&&0&-1\\0&&&&&1&0\end{pmatrix}. \tag{A1.3}$$

Now it is convenient to separate the variables that will be integrated from those that will not. With this purpose we define the vectors

$$\boldsymbol{R}'_a=(x_{a1},y_{a1},\ldots,x_{ak},y_{ak}),\qquad \boldsymbol{R}''_a=(x_{a,k+1},y_{a,k+1},\ldots,x_{an},y_{an}), \tag{A1.4}$$

such that $\boldsymbol{R}_a=(\boldsymbol{R}'_a,\boldsymbol{R}''_a)$ and $\boldsymbol{M}\boldsymbol{R}_a=(\boldsymbol{M}'\boldsymbol{R}'_a,\boldsymbol{M}''\boldsymbol{R}''_a)$. $\boldsymbol{M}'$ and $\boldsymbol{M}''$ are defined as in (A1.3), and have dimensionalities $2k$ and $2(n-k)$, respectively. Thus we have

$$p_0\begin{pmatrix}\boldsymbol{R}'_1,\boldsymbol{R}''_1\\\boldsymbol{R}'_2,\boldsymbol{R}''_2\end{pmatrix}=\delta(1-|\boldsymbol{R}'_1|^2-|\boldsymbol{R}''_1|^2)\delta(1-|\boldsymbol{R}'_2|^2-|\boldsymbol{R}''_2|^2)$$
$$\times\delta(\boldsymbol{R}'_1\cdot\boldsymbol{R}'_2+\boldsymbol{R}''_1\cdot\boldsymbol{R}''_2)\delta(\boldsymbol{R}'_1\cdot\boldsymbol{M}'\boldsymbol{R}'_2+\boldsymbol{R}''_1\cdot\boldsymbol{M}\boldsymbol{R}''_2). \tag{A1.5}$$

In order to integrate over $\boldsymbol{R}_1''$ and $\boldsymbol{R}_2''$ we write vectors in spherical coordinates, i.e.

$$p_0\begin{pmatrix}\boldsymbol{R}_1'\\ \boldsymbol{R}_2'\end{pmatrix}=\int \delta(a_1^2-R_1''^2)\delta(a_2^2-R_2''^2)\delta(\alpha+\boldsymbol{R}_1''\cdot\boldsymbol{R}_2'')\delta(\beta+\boldsymbol{R}_1''\cdot\boldsymbol{M}''\boldsymbol{R}_2'')$$
$$\times R_1''^{2m-1}\,\mathrm{d}R_1''\,\mathrm{d}^{(2m)}\Omega_1\,R_2''^{2m-1}\,\mathrm{d}R_2''\,\mathrm{d}^{(2m)}\Omega_2 \tag{A1.6}$$

where

$$\begin{aligned}
&a_a^2=1-R_a'^2, \qquad m=n-k,\\
&\alpha=\boldsymbol{R}_1'\cdot\boldsymbol{R}_2', \qquad \beta=\boldsymbol{R}_1'\cdot\boldsymbol{M}'\boldsymbol{R}_2',\\
&\mathrm{d}^{(2m)}\Omega_a=\sin^{2m-2}\theta_a\,\mathrm{d}\theta_a\,\sin^{2m-3}\varphi_{a1}\,\mathrm{d}\varphi_{a1}\ldots\\
&\qquad\equiv\sin^{2m-2}\theta_a\,\mathrm{d}\theta_a\,\mathrm{d}\omega_a.
\end{aligned} \tag{A1.7}$$

In the procedure of integration we start by evaluating the integral

$$I=\int \delta(\alpha-\boldsymbol{a}\cdot\boldsymbol{r})\delta(\beta-\boldsymbol{b}\cdot\boldsymbol{r})\,\mathrm{d}^{(m)}\Omega_r \tag{A1.8}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are any two fixed vectors and $\boldsymbol{r}$ is an m-dimensional real vector whose orientation is integrated over. It is convenient to align the vector $\boldsymbol{a}$ with the 'z axis', so that, in spherical coordinates,

$$\begin{aligned}
&\boldsymbol{a}=a(1,0,\ldots,0), \qquad \boldsymbol{b}=b(\cos\theta_0,\sin\theta_0,0,\ldots,0),\\
&\boldsymbol{r}=r(\cos\theta,\sin\theta\cos\varphi_1,\ldots,\sin\varphi_{m-2}), \qquad \mathrm{d}^{(m)}\Omega_r=\sin^{m-2}\theta\,\mathrm{d}\theta\,\mathrm{d}\omega.
\end{aligned} \tag{A1.9}$$

We obtain for I the following result:

$$I=\frac{\sin^{3-m}\theta_0}{abr^2}\left[1-\left(\frac{\alpha}{ar}\right)^2-\left(\frac{\beta}{br}\right)^2-\frac{2\alpha\beta}{abr^2}\cos\theta_0\right]^{(m-4)/2}.$$

Using this result to integrate over Ω_1 in equation (A1.6) we have

$$p_0\begin{pmatrix}\boldsymbol{R}_1'\\ \boldsymbol{R}_2'\end{pmatrix}\propto\int \delta(a_1^2-R_1''^2)\delta(a_2^2-R_2''^2)\frac{\sin^{3-2m}(\boldsymbol{R}_2'',\boldsymbol{M}''\boldsymbol{R}_2'')}{R_2''|\boldsymbol{M}''\boldsymbol{R}_2|^2R_1''^2}$$
$$\times\left[1-\left(\frac{\alpha}{R_1''R_2''}\right)^2-\left(\frac{\beta}{|\boldsymbol{M}\boldsymbol{R}_2''|R_1''}\right)^2-\frac{2\alpha\beta\cos(\boldsymbol{R}_2'',\boldsymbol{M}''\boldsymbol{R}_2'')}{R_1''^2R_2''|\boldsymbol{M}''\boldsymbol{R}_2''|}\right]^{m-2}$$
$$\times R_1''^{2m-1}R_2''^{2m-1}\,\mathrm{d}R_1''\,\mathrm{d}R_2''\,\mathrm{d}^{(2m)}\Omega_2. \tag{A1.10}$$

Here it can be seen that

$$\cos(\boldsymbol{R}_2'',\boldsymbol{M}''\boldsymbol{R}_2'')=\boldsymbol{R}_2''\cdot\boldsymbol{M}''\boldsymbol{R}_2''/|\boldsymbol{R}_2''|\,|\boldsymbol{M}''\boldsymbol{R}_2''|=0 \tag{A1.11}$$

and the integral takes the form

$$p_0\begin{pmatrix}\boldsymbol{R}_1'\\ \boldsymbol{R}_2'\end{pmatrix}\propto\int \delta(a_1^2-R_1''^2)\delta(a_2^2-R_2''^2)R_1''^{2m-3}R_2''^{2m-3}\left(1-\frac{\alpha^2+\beta^2}{R_1''^2R_2''^2}\right)^{m-2}\mathrm{d}R_1''\mathrm{d}R_2'' \tag{A1.12a}$$
$$\propto(a_1^2a_2^2-\alpha^2-\beta^2)^{m-2}\vartheta(a_1^2)\vartheta(a_2^2). \tag{A1.12b}$$

$\vartheta(a^2)$ is the usual step function, which appears as a consequence of unitarity. Finally, to write the result in a more transparent form, we define the complex vectors

$$\boldsymbol{r}_1=(U_{11},\ldots,U_{1k}), \qquad \boldsymbol{r}_2=(U_{21},\ldots,U_{2k}), \tag{A1.13}$$

in terms of which we have

$$p_0(\boldsymbol{r}_1, \boldsymbol{r}_2) \propto \vartheta(1-|\boldsymbol{r}_1|^2)\vartheta(1-|\boldsymbol{r}_2|^2)[1-|\boldsymbol{r}_1|^2-|\boldsymbol{r}_2|^2+|\boldsymbol{r}_1|^2|\boldsymbol{r}_2|^2-|\boldsymbol{r}_1\cdot\boldsymbol{r}_2^*|^2]^{m-2}, \tag{A1.14}$$

which is equation (2.7) of the text.

Appendix 2. Proof of equation (3.12)

The probability density $p_0(S_{11})$ is given by

$$p_0(S_{11}) \propto \int \delta\Big(S_{11} - \sum_{a=1}^{n} (U_{1a})^2\Big)\delta\Big(1-\sum_{a=1}^{n}|U_{1a}|^2\Big)\,\mathrm{d}U_{11}\ldots\mathrm{d}U_{1n},$$
$$U_{ab} = x_{ab} + \mathrm{i}y_{ab}, \qquad \mathrm{d}U_{ab} = \mathrm{d}x_{ab}\,\mathrm{d}y_{ab}. \tag{A2.1}$$

We define the vectors

$$\boldsymbol{x}_1 = (x_{11}, x_{12}, \ldots, x_{1n}), \qquad \boldsymbol{y}_1 = (y_{11}, y_{12}, \ldots, y_{1n}), \tag{A2.2}$$

and express them in spherical coordinates, to have

$$p_0(S_{11}) = \int \delta(X_{11} - x_1^2 + y_1^2)\delta(Y_{11} - 2\boldsymbol{x}_1\cdot\boldsymbol{y}_1)\delta(1-x_1^2-y_1^2)$$
$$\times x_1^{n-1}\,\mathrm{d}x_1 \sin^{n-2}\theta_1\,\mathrm{d}\theta_1\,\mathrm{d}\omega_1 y_1^{n-1}\,\mathrm{d}y_1\sin^{n-2}\theta_2\,\mathrm{d}\theta_2\,\mathrm{d}\omega_2 \tag{A2.3}$$

where $S_{11} = X_{11} + \mathrm{i}Y_{11}$ and $\mathrm{d}\omega_i$ is a shorthand notation to symbolise the remaining angular part.

The integration can be carried out directly with the same procedure used in appendix 1 of fixing one of the vectors along the 'z axis', while we integrate over the other, in such a way that the relative angle $(\boldsymbol{x}_1, \boldsymbol{y}_1)$ can be identified with the corresponding angle θ_i. We obtain

$$p_0(S_{11}) \propto (1 - X_{11}^2 - Y_{11}^2)^{(n-3)/2} = (1-|S_{11}|^2)^{(n-3)/2}.$$

Appendix 3. Proof of equation (3.23)

To obtain the joint probability density for the four S-matrix elements $\{S_{11}, S_{12}, S_{21}, S_{22}\}$, we must carry out the following integration:

$$p_0\begin{pmatrix} S_{11}, S_{12} \\ S_{21}, S_{22}\end{pmatrix} = \int \delta\Big(S_{11} - \sum_c (U_{1c})^2\Big)\delta\Big(S_{12} - \sum_c U_{1c}U_{2c}\Big)$$
$$\times\delta\Big(S_{21} - \sum_c U_{2c}U_{1c}\Big)\delta\Big(S_{22} - \sum_c (U_{2c})^2\Big)\delta\Big(1-\sum_c |U_{1c}|^2\Big)$$
$$\times\delta\Big(1-\sum_c|U_{2c}|^2\Big)\delta\Big(\sum_c (x_{1c}x_{2c} + y_{1c}y_{2c})\Big)$$
$$\times\delta\Big(\sum_c (x_{1c}y_{2c} - x_{2c}y_{1c})\Big)\,\mathrm{d}U_{11}\ldots\mathrm{d}U_{2n} \tag{A3.1}$$

where again $U_{ab} = x_{ab} + \mathrm{i}y_{ab}$. Defining the vectors

$$\begin{aligned} \boldsymbol{x}_a &= (x_{a1}, x_{a2}, \ldots, x_{an}), \\ \boldsymbol{y}_a &= (y_{a1}, y_{a2}, \ldots, y_{an}), \end{aligned} \qquad a = 1, 2, \tag{A3.2}$$

and expressing them in spherical coordinates, we write the joint probability density as

$$\begin{aligned} p_0\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \int & \delta(Y_{11} - 2\boldsymbol{x}_1 \cdot \boldsymbol{y}_1)\delta(X_{12} - \boldsymbol{x}_1 \cdot \boldsymbol{x}_2 + \boldsymbol{y}_1 \cdot \boldsymbol{y}_2) \\ & \times \delta(Y_{12} - \boldsymbol{y}_1 \cdot \boldsymbol{x}_2 - \boldsymbol{x}_1 \cdot \boldsymbol{y}_2)\delta(X_{21} - \boldsymbol{x}_1 \cdot \boldsymbol{x}_2 + \boldsymbol{y}_1 \cdot \boldsymbol{y}_2) \\ & \times \delta(Y_{21} - \boldsymbol{y}_1 \cdot \boldsymbol{x}_2 - \boldsymbol{x}_1 \cdot \boldsymbol{y}_2)\delta(Y_{22} - 2\boldsymbol{x}_2 \cdot \boldsymbol{y}_2)\delta(\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 + \boldsymbol{y}_1 \cdot \boldsymbol{y}_2) \\ & \times \delta(\boldsymbol{x}_1 \cdot \boldsymbol{y}_2 - \boldsymbol{y}_1 \cdot \boldsymbol{x}_2)\delta(X_{11} - x_1^2 + y_1^2)\delta(X_{22} - x_2^2 + y_2^2) \\ & \times \delta(1 - x_1^2 - y_1^2)\delta(1 - x_2^2 - y_2^2) x_1^{n-1}\, \mathrm{d}x_1\, \mathrm{d}^{(n)}\Omega_{x_1} \ldots y_2^{n-1}\, \mathrm{d}y_2\, \mathrm{d}^{(n)}\Omega_{y_2}. \end{aligned} \tag{A3.3}$$

Now we start by integrating the angular part of $\boldsymbol{y}_2$. For this, we select all those delta functions whose argument contains a scalar product of $\boldsymbol{y}_2$ with the other vectors. We thus have the integral

$$\begin{aligned} I \equiv \int & \delta(X_{12} - \boldsymbol{x}_1 \cdot \boldsymbol{x}_2 + \boldsymbol{y}_1 \cdot \boldsymbol{y}_2)\delta(Y_{12} - \boldsymbol{y}_1 \cdot \boldsymbol{x}_2 - \boldsymbol{x}_1 \cdot \boldsymbol{y}_2) \\ & \times \delta(X_{21} - \boldsymbol{x}_1 \cdot \boldsymbol{x}_2 + \boldsymbol{y}_1 \cdot \boldsymbol{y}_2)\delta(Y_{21} - \boldsymbol{y}_1 \cdot \boldsymbol{x}_2 - \boldsymbol{x}_1 \cdot \boldsymbol{y}_2) \\ & \times \delta(Y_{22} - 2\boldsymbol{x}_2 \cdot \boldsymbol{y}_2)\delta(\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 + \boldsymbol{y}_1 \cdot \boldsymbol{y}_2)\delta(\boldsymbol{x}_1 \cdot \boldsymbol{y}_2 - \boldsymbol{y}_1 \cdot \boldsymbol{x}_2)\, \mathrm{d}^{(n)}\Omega_{y_2}. \end{aligned} \tag{A3.4}$$

Let

$$\begin{aligned} &\rho = Y_{11}, && \alpha = X_{12} - \boldsymbol{x}_1 \cdot \boldsymbol{x}_2, && \beta = Y_{12} - \boldsymbol{y}_1 \cdot \boldsymbol{x}_2, \\ &\gamma = Y_{22}, && \sigma = X_{21} - \boldsymbol{x}_1 \cdot \boldsymbol{x}_2, && \tau = Y_{21} - \boldsymbol{y}_1 \cdot \boldsymbol{x}_2, \\ &\mu = \boldsymbol{y}_1 \cdot \boldsymbol{x}_2, && \nu = \boldsymbol{x}_1 \cdot \boldsymbol{x}_2. \end{aligned} \tag{A3.5}$$

It is evident that the integral I reduces to

$$\begin{aligned} I = & \delta(\sigma - \alpha)\delta(\tau - \beta)\delta(\nu - \alpha)\delta(\mu - \beta) \\ & \times \int \delta(\alpha + \boldsymbol{y}_1 \cdot \boldsymbol{y}_2)\delta(\beta - \boldsymbol{x}_1 \cdot \boldsymbol{y}_2)\delta(\gamma - 2\boldsymbol{x}_2 \cdot \boldsymbol{y}_2)\, \mathrm{d}^{(m)}\Omega_{y_2}, \end{aligned} \tag{A3.6}$$

or equivalently to

$$\begin{aligned} I = & \delta(S_{12} - S_{21})\delta(\nu - \alpha)\delta(\mu - \beta) \\ & \times \int \delta(\alpha + \boldsymbol{y}_1 \cdot \boldsymbol{y}_2)\delta(\beta - \boldsymbol{x}_1 \cdot \boldsymbol{y}_2)\delta(\gamma - 2\boldsymbol{x}_1 \cdot \boldsymbol{y}_2)\, \mathrm{d}^{(m)}\Omega_{y_2}. \end{aligned} \tag{A3.7}$$

Clearly, the first delta is a consequence of the S-matrix symmetry. Then, to obtain the integral I we need the following integral,

$$G_1 = \int \delta(\alpha + \boldsymbol{a} \cdot \boldsymbol{r})\delta(\beta + \boldsymbol{b} \cdot \boldsymbol{r})\delta(\gamma + \boldsymbol{c} \cdot \boldsymbol{r})\, \mathrm{d}^{(m)}\Omega_r, \tag{A3.8}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are any fixed vectors and $\boldsymbol{r}$ is an m-dimensional real vector whose angular part is integrated. Fixing $\boldsymbol{a}$ along the 'z axis' $(1, 0 \ldots 0)$, the vector $\boldsymbol{b}$ in the plane defined by the 'z axis' and the vector $(0, 1, 0 \ldots 0)$ and $\boldsymbol{c}$ with components along $(1, 0, 0, \ldots)$, $(0, 1, 0, \ldots)$ and $(0, 0, 1, 0, \ldots)$, we have

$$\begin{aligned}
&\boldsymbol{a} = a(1, 0, \ldots, 0), \qquad \boldsymbol{b} = b(\cos\theta_{ab}, \sin\theta_{ab}, 0, \ldots, 0),\\
&\boldsymbol{c} = c(\cos\theta_{ac}, \sin\theta_{ac}\cos\varphi_c, \sin\theta_{ac}\sin\varphi_c, 0, \ldots, 0),\\
&\boldsymbol{r} = r(\cos\theta, \sin\theta\sin\varphi, \sin\theta\sin\varphi\cos\psi, \ldots),\\
&\mathrm{d}^{(m)}\Omega_r = \sin^{m-2}\theta \sin^{m-3}\varphi \sin^{m-4}\psi \,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}\psi\,\mathrm{d}\omega.
\end{aligned} \tag{A3.9}$$

As in appendix 2, $\mathrm{d}\omega$ is the remaining angular part whose integral is a constant. The integration is direct and gives for G_1 the following result:

$$G_1 \propto \frac{[(1-x^2)(1-y^2)(1-z^2)]^{(m-5)/2}}{abcr^3 \sin\theta_{ab}\sin\theta_{ac}\sin\varphi_c} \tag{A3.10}$$

with

$$\begin{aligned}
&x = -\frac{\alpha}{ar}, \qquad y = -\frac{\beta + brx\cos\theta_{ab}}{(1-x^2)^{1/2} br \sin\theta_{ab}},\\
&z = -\frac{\gamma + crx\cos\theta_{ab} + cry(1-x^2)^{1/2}\sin\theta_{ac}\cos\theta_{ac}}{(1-x^2)^{1/2}(1-y^2)^{1/2} cr \sin\theta_{ac}\sin\varphi_c}.
\end{aligned} \tag{A3.11}$$

If we use this result to evaluate the integral that appears in equation (A3.7), where we choose

$$\begin{aligned}
&\boldsymbol{x}_1 = x_1(1, 0, \ldots, 0), \qquad \boldsymbol{y}_1 = y_1(\cos\theta_1, \sin\theta_1, 0, \ldots, 0),\\
&\boldsymbol{x}_2 = x_2(\cos\theta_2, \sin\theta_2\cos\varphi_2, \sin\theta_2\sin\varphi_2, 0, \ldots, 0),\\
&\boldsymbol{y}_2 = y_2(\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi\cos\psi, \ldots),
\end{aligned} \tag{A3.12}$$

I becomes

$$I \propto \frac{\delta(S_{12}-S_{21})\delta(\nu-\alpha)\delta(\mu-\beta)}{x_1 y_1 x_2 y_2^3 \sin^{n-4}\theta_1 \sin^{n-4}\theta_2 \sin^{n-4}\varphi_2} J(\theta_1, \theta_2, \varphi_2, x_1, x_2, y_1, y_2) \tag{A3.13}$$

with

$$\begin{aligned}
J(\theta_1, \ldots, y_2) = {} & 1 - \cos^2\theta_1 - \cos^2\theta_2 - \cos^2\varphi_2 + \cos^2\theta_1\cos^2\theta_2 + \cos^2\theta_1\cos^2\varphi_2\\
& + \cos^2\theta_2\cos^2\varphi_2 - \cos^2\theta_1\cos^2\theta_2\cos^2\varphi_2 - (\alpha/y_1y_2)^2(1-\cos^2\theta_2)\\
& - (\beta/x_1y_2)^2(1-\cos^2\varphi_2) - (\gamma/x_2y_2)^2(1-\cos^2\theta_1)\\
& + (\beta/x_1y_2)^2(\cos^2\theta_1\cos^2\theta_2 - \cos^2\theta_1\cos^2\varphi_2\\
& - \cos^2\theta_2\cos^2\varphi_2 + \cos^2\theta_1\cos^2\theta_2\cos^2\varphi_2)\\
& - (2\alpha\beta/x_1y_1y_2^2)(\cos\theta_1 - \cos\theta_1\cos^2\theta_2 - \sin\theta_1\sin\theta_2\cos\theta_2\cos\varphi_2)\\
& + (\gamma\alpha/x_1x_2y_2^2)(\cos\theta_2 - \cos\theta_2\cos^2\theta_1 - \cos\theta_1\sin\theta_2\sin\theta_1\cos\varphi_2)\\
& - (\mu\beta/y_1x_2y_2^2)\sin\theta_1\sin\theta_2\cos\varphi_2\\
& + (2\alpha/x_1y_2)\sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2\cos\varphi_2.
\end{aligned} \tag{A3.14}$$

Once we have the function I, we need, as an intermediate step towards (A3.3), to evaluate the integral

$$I_1 \propto \delta(S_{12}-S_{21}) \int \delta(Y_{11}-2\boldsymbol{x}_1\cdot\boldsymbol{y}_1)\delta(X_{12}-2\boldsymbol{x}_1\cdot\boldsymbol{x}_2)\delta(Y_{12}-\boldsymbol{y}_1\cdot\boldsymbol{x}_2)$$

$$\times \frac{J(\theta_1,\theta_2,\ldots,y_2)}{x_1 y_1 x_2 y_2^3 \sin^{n-4}\theta_1 \sin^{n-4}\theta_2 \sin^{n-4}\varphi_2}\, \mathrm{d}^{(n)}\Omega_{y_1}\, \mathrm{d}^{(n)}\Omega_{x_1}\, \mathrm{d}^{(n)}\Omega_{x_2} \qquad \text{(A3.15)}$$

which gives

$$I_1 \propto [\delta(S_{12}-S_{21})/x_1^3 y_1^3 x_2^3 y_2^3] H(x_1,\ldots,y_2,X_{11},\ldots,Y_{22}) \qquad \text{(A3.16)}$$

with

$$H(x_1,\ldots,Y_{22}) = \Bigg[1-\left(\frac{Y_{11}}{2x_1y_1}\right)^2-\left(\frac{X_{12}}{2x_1x_2}\right)^2-\left(\frac{Y_{12}}{2x_1y_2}\right)^2-\left(\frac{Y_{12}}{2y_1y_2}\right)^2-\left(\frac{X_{12}}{2y_1y_2}\right)^2$$

$$-\left(\frac{Y_{22}}{2x_2y_2}\right)^2+\frac{Y_{12}Y_{11}X_{12}}{4x_1^2y_1^2x_2^2}-\frac{X_{12}Y_{12}Y_{11}}{4x_1^2y_1^2y_2^2}+\frac{Y_{12}Y_{22}X_{12}}{4x_1^2x_2^2y_2^2}$$

$$-\frac{Y_{12}Y_{22}X_{12}}{4y_1^2x_2^2y_2^2}+\left(\frac{1}{4x_1y_1x_2y_2}\right)^2(Y_{12}^4+X_{12}^4+Y_{11}^2Y_{22}^2$$

$$+2X_{12}^2Y_{11}Y_{22}-2Y_{11}Y_{12}^2Y_{22}+2X_{12}^2Y_{12}^2)\Bigg]^{(n-5)/2}. \qquad \text{(A3.17)}$$

Finally, making the integration over the magnitudes, we have, after some algebra, the followiing result for the four-element joint probability density:

$$p_0\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22}\end{pmatrix} \propto \delta(S_{12}-S_{21})[1-|S_{11}|^2-|S_{22}|^2-2|S_{12}|^2+|S_{11}S_{22}-(S_{12})^2|^2]^{(n-5)/2}, \qquad \text{(A3.18)}$$

which is equation (3.23).

References

Agassi D, Weidenmüller H A and Mantzouranis G 1975 *Phys. Rep.* **22** 145
De los Reyes J, Mello P A and Seligman T H 1980 *Z. Phys.* **295** 247
Dyson F J 1962 *J. Math. Phys.* **3** 140
Gaudin M and Mello P A 1981 *J. Phys. G: Nucl. Phys.* **7** 1085
Hauser W and Feshbach H 1952 *Phys. Rev.* **87** 366
Hofmann H M, Mertelmeier T, Seligman T H and Mello P A 1981 *Nucl. Phys.* **A372** 185
Hofmann H M, Richert J, Tepel J W and Weidenmüller H A 1975 *Ann. Phys., NY* **90** 403
Kawai M, Kerman A K and McVoy K W 1973 *Ann. Phys., NY* **75** 156
Mello P A 1979 *Phys. Lett.* **81B** 103
Mello P A and Seligman T H 1980 *Nucl. Phys.* **A344** 489
Moldauer P A 1975 *Phys. Rev.* **C12** 744
—— 1978 *Statistical Theory of neutron nuclear reactions, Course on nuclear theory for applications, at the International Centre for Theoretical Physics. Trieste, Italy*